

Flowing Sand—A Possible Physical Realization of Directed Percolation

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A simple model for flowing sand on an inclined plane is introduced. The model is related to recent experiments by Douady and Daerr and reproduces some of the experimentally observed features. Avalanches of intermediate size appear to be compact, placing the critical behavior of the model into the universality class of compact directed percolation. On very large scales, however, the avalanches break up into several branches, leading to a crossover from compact to ordinary directed percolation. Thus, systems of flowing granular matter on an inclined plane could serve as a first physical realization of directed percolation.

KEY WORDS: Directed percolation; nonequilibrium phase transitions; sandpiles.

I. INTRODUCTION

Directed Percolation (DP) is probably the simplest model that exhibits a non-equilibrium phase transition between an active “wet” phase and an inactive “dry” one.⁽¹⁾ In the dry phase the system is in a single “absorbing” state from where it cannot escape. Interest in DP mainly stems from *universality* of the associated critical behavior. It is believed that transitions in all models with an absorbing state belong to the DP universality class (unless there are some special underlying symmetries). DP exponents were

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measured for an extremely wide variety of models. Even though the exponents have not yet been calculated analytically, their values (especially in $1 + 1$ dimensions) are known with very high precision.⁽²⁾

Despite the preponderance of models in the DP universality class, so far no physical system has been found to exhibit DP behavior. Indeed, as noted by Grassberger,

"...there is still no experiment where the critical behavior of DP was seen. This is a very strange situation in view of the vast and successive theoretical efforts made to understand it. Designing and performing such an experiment has thus top priority in my list of open problems."⁽³⁾

The purpose of this paper is to point out that a simple system of flowing sand on an inclined plane, that has recently been introduced and studied by Daerr and Douady (DD), may well be the first physical realization of a transition in the DP universality class.⁽⁴⁻⁶⁾ In Section II we describe these experiments in fair detail. The data presented by DD raises serious questions regarding the applicability of DP. In particular, the observed shapes of wet clusters differ from those seen in standard DP simulations; they are much more compact. Since the corresponding model, called Compact Directed Percolation (CDP), is unstable against perturbations towards the standard DP behavior,⁽⁷⁾ the latter is the generic case expected to occur (if no parameters were fine-tuned to place the system in the CDP class).

This motivated us to look for a simple model which is defined in terms of dynamic rules that can plausibly be related to the experiments and, at the same time, exhibit features that look like the experimentally observed ones. Whether the transition exhibited by such a model does belong to the DP universality class remains to be investigated.

Such a model is introduced in Section III. It is a *directed sandpile* model, which is simpler than the one introduced and analyzed by Tadic and Dhar,⁽⁸⁾ here the system is reset to a uniform initial state after each avalanche. In Section IV we show the outcome of some simulations. The avalanches (observed in the active phase) reproduce the experimental observations quite well. We establish the existence of a transition from an active to an inactive phase. However, the critical behavior extracted from these figures does not seem to be in the DP universality class, rather, it seems close to CDP. As it turns out, this CDP type critical behavior is only a transient: the true critical behavior *is* of the DP type, but can only be seen after a very long crossover regime, in which the exponents are those of CDP. This observation is based on a careful numerical study, which is presented in Sections V and VI.

Our conclusion is that the DD experiment does serve as a possible realization of a DP-type transition. Observation of DP exponents may be

tricky as a substantial crossover regime may mask the true critical behavior, and one should try to find methods to shorten this regime.

Finally we should note that the preparation procedure of the DD system can be considered a simple case of Self Organized Criticality (SOC).⁽⁹⁾ As in the SOC sandpile model introduced by Tadic and Dhar,⁽⁸⁾ the system “prepares itself” at the critical point of a DP type transition without any fine tuning. However, in the present case the driving mechanism is slightly different. While in SOC models a slow driving force causes the evolution into a critical state, avalanches in the DD experiment are started individually by hand.

II. THE DOUADY-DAERR EXPERIMENT

The experimental apparatus consists of an inclined plane (size of about 1m) covered by a rough velvet cloth; the angle of inclination φ_0 can be varied. Glass beads (e.g., “sand”) of diameter 250–425 μm ⁽⁴⁾ are poured uniformly at the top of the plane and flow down while a thin layer of thickness $h = h_d(\varphi_0)$, consisting of several monolayers, settles and remains immobile. At this thickness the sand is *dynamically stable*; the value of h_d decreases with an increasing angle of inclination.

For each φ_0 there exists another thickness h_s with $h_s(\varphi_0) > h_d(\varphi_0)$, beyond which a *static* layer becomes unstable. Hence there exists a region (see Fig. 1) in the (φ, h) plane, in which a static layer is stable but a flowing one is unstable. We can now take the system, that settled at $h_d(\varphi_0)$, and increase its angle of inclination to φ , staying within this region of mixed stability. The layer will not flow spontaneously, but if we disturb it at the top, generating a flow near the perturbation, the flow will persist and an avalanche will be generated, leaving behind a layer of thickness $h_d(\varphi)$. These avalanches had the shape of a fairly regular triangle with opening angle θ . As the increment of the inclination

$$\Delta\varphi = \varphi - \varphi_0$$

decreases, the value of $\theta(\Delta\varphi)$ decreases as well and the area affected by the avalanche decreases, vanishing as $\Delta\varphi \rightarrow 0$. This calls for testing a power law behavior of the form

$$\theta \sim (\Delta\varphi)^x \quad (1)$$

If instead of increasing φ we lower the plane, i.e., go to $\Delta\varphi < 0$, our system, whose thickness is $h_d(\varphi_0)$, is below the present thickness of dynamic stability, $h_d(\varphi)$. We believe that in this case an initial perturbation will not

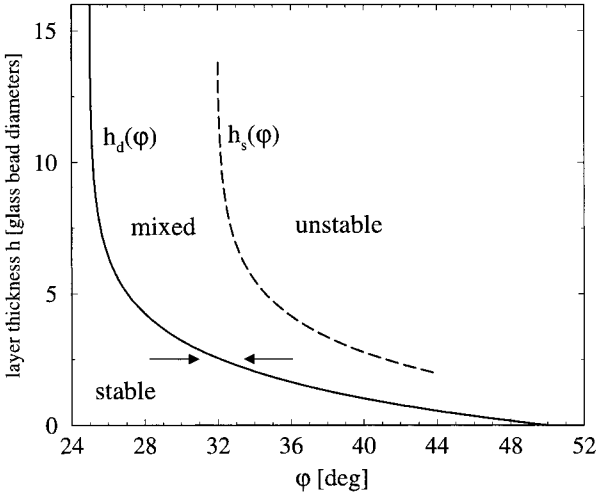


Fig. 1. Schematic stability diagram of the DD-experiment. A layer of thickness h is dynamically stable below a certain threshold $h_d(\varphi)$ (solid line). Due to friction forces non-moving layers remain stable in the mixed region below static stability limit $h_s(\varphi)$ (dashed line). In the present work we investigate the properties in the vicinity of the dynamic phase transition line, as indicated by the arrows.

propagate, it will rather die out after a certain time (or beyond a certain size $\xi_{||}$ of the transient avalanche). As the deviation $|\Delta\varphi|$ decreases, we expect the size of the transient active region to increase, i.e., the decay length should grow according to a power law

$$\xi_{||} \sim (-\Delta\varphi)^{-\nu_{||}} \quad (2)$$

Hence, by pouring sand at inclination φ_0 , DD produced a *self-organized critical system*. The system is precisely at the borderline (with respect to changing the angle) between a stable regime $\varphi < \varphi_0$ in which perturbations die out and an unstable one, $\varphi > \varphi_0$, where perturbations persist and spread. This kind of self-organized criticality during the preparation procedure differs from standard SOC models⁽⁹⁾ in which a slow driving force (acting on a time scale much smaller than that of the system's dynamic response) causes evolution to a critical state.

Once this connection has been made, it is natural to associate this system with the problem of DP. Denote by p either the site or bond percolation probability and by p_c its critical value (i.e., for $p > p_c$ the system is in the active phase). We associate the change in tilt with $p - p_c$, assuming

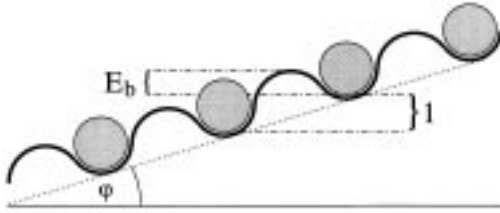


Fig. 2. The top layer of sand in an effective washboard potential.

that near the angle of preparation the behavior of the sand system is related to a DP problem with

$$\Delta\varphi = \varphi - \varphi_0 \propto p - p_c \tag{3}$$

Hence, the exponent ν_{\parallel} should be compared with the known values for DP and CDP. The exponent x in Eq. (1) can also be measured and compared with

$$\tan \theta \sim \xi_{\perp} / \xi_{\parallel} \sim (\Delta\varphi)^{\nu_{\parallel} - \nu_{\perp}} \tag{4}$$

III. THE MODEL

Our aim is to write down a simple model based on the physics of flowing sand. We adopt the observation made by DD, that in the regime of interest (i.e., for tilt angles close to φ_0) grains of the top layer of sand rest on grains of the layer below (rather than on other grains of the top layer).⁴ Hence the lower layers provide for the top one a kind of *washboard potential*, as depicted in Fig. 2.

We further assume that only the top layer participates in an avalanche and therefore place the grains of this layer on the sites of a regular square lattice⁵ (see Fig. 3). At any given time a particular horizontal row of grains may become active, while at the next time step the activity may be transferred to the row beneath. The physical picture that underlies the model is as follows. A grain G may become active if at least one of the neighboring grains in the row above it has been active at the previous time step. These grains may then transfer energy to G ; if $\Delta E(G)$, the total energy transferred to G , exceeds the barrier E_b of the washboard, G becomes active. An active grain “rolls down” at the next time step and collides with the grains of the next row. The energy it brings to these collisions is $1 + \Delta E(G)$, where 1 is the potential energy due to the height difference between two consecutive

⁴ This holds for $\varphi < \varphi_0$ and also for $\varphi > \varphi_0$, as long as we stay within the region of mixed stability.

⁵ We chose to work with a square lattice, but could have used a triangular one as well, with each site communicating with two neighbors above and two below.

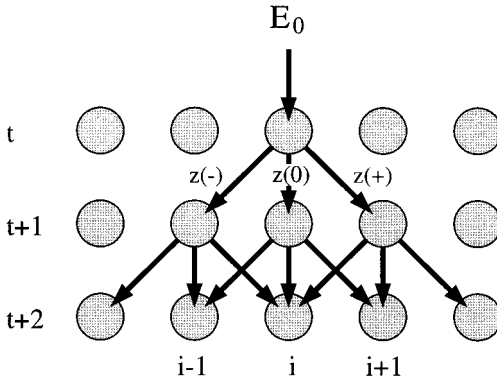


Fig. 3. Energy transfer between grains on a square lattice.

rows. A fraction f of its total energy is dissipated, while the rest is divided stochastically among its three neighbors from the lower row.

The model is hence defined in terms of two variables; an activation variable,

$$S_i^t = \begin{cases} 1 & \text{if grain } (t, i) \text{ active,} \\ 0 & \text{otherwise,} \end{cases}$$

and an energy variable E_i^t . The index t denotes rows of our square lattice *and* time; at time t we update the states of the grains belonging to row t . Energy is measured in units of the difference between two successive minima of the potential (see Fig. 2). The model is controlled by two parameters, namely

- E_b , the barrier height, and
- f , the fraction of dissipated energy.

The dynamic rules of our model are defined in terms of these variables and parameters as follows. For given values of activities S_i^t and energies E_i^t we first calculate the energy transferred to the grains of the next row $t+1$. To this end we generate for each active site $S_i^t=1$ three random numbers, $z_i^t(\delta)$ (with $\delta = \pm 1, 0$) in a way that

$$\sum_{\delta = \pm 1, 0} z_i^t(\delta) = 1 \quad (5)$$

The energy transferred to grain $(t + 1, i)$ is then given by

$$\Delta E_i^{t+1} = (1 - f) \sum_{\delta = \pm 1, 0} S_{i-\delta}^t E_{i-\delta}^t z_{i-\delta}^t(\delta) \tag{6}$$

The values of these energies determine the activation of the grains of row $t + 1$:

$$S_i^{t+1} = \begin{cases} 1 & \text{if } \Delta E_i^{t+1} > E_b, \\ 0 & \text{if } \Delta E_i^{t+1} \leq E_b \end{cases} \tag{7}$$

The energies of the active grains are set according to

$$E_i^{t+1} = S_i^{t+1}(1 + \Delta E_i^{t+1}) \tag{8}$$

The meaning of these rules, in words, is obvious: the energy of site i at time $t + 1$ is obtained by identifying, among its three neighbors of the preceding row, those sites (or grains) that were active at time t . At each such active site (t, i) we generated three random numbers $z_i^t(\delta)$ which represent the fraction of energy transferred from the grain at site (t, i) to the one at $(t + 1, i + \delta)$. We add up the energy contributions from these active sites; the fraction $1 - f$ is *not* dissipated and compared to the barrier height E_b . If the acquired energy ΔE_i^{t+1} exceeds E_b , site $(t + 1, i)$ becomes active, rolls over the barrier bringing to the collisions (at time $t + 2$) the acquired energy calculated above *and* its excess potential energy (of value 1).

IV. SHORT-TIME SIMULATIONS AND QUALITATIVE DISCUSSION OF THE TRANSITION

Let us consider the behavior of our model as we vary E_b at a fixed value of the dissipation. We expect that for small values of E_b an active grain will activate the grains below with high probability; avalanches will propagate downhill and also spread sideways. For a strongly localized initial activation we should, therefore, observe activated regions of triangular shape. As E_b increases, the rate of activation decreases and the opening angle θ of these triangles should decrease, until E_b reaches a critical value E_b^c , beyond which initial activations die out in a finite number of time steps (or rows). These expectations are indeed borne out by simulations of the model: the critical value E_b^c depends on the dissipation f and the resulting phase transition line is shown in Fig. 4 as a solid line.

In order to understand this transition qualitatively, let us consider a simple mean-field type approximation, in which all stochastic variables are replaced by their average values.

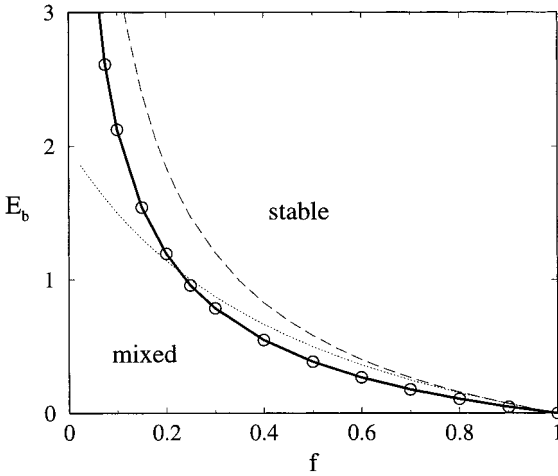


Fig. 4. Phase diagram of the model for flowing sand. The full line represents the phase transition line. The mean-field approximations of Eqs. (10) and (14) are shown as dotted and dashed lines, respectively.

We consider an edge separating an active region from an inactive one at time t : sites to the left of i and i itself are wet, whereas $i+1, i+2, \dots$ are dry. Will the rightmost wet site be wet or dry at the next time step? Assuming that all wet sites at time t have the same energy E^t , in our mean-field type estimate the energy delivered to site i at time $t+1$ is

$$\Delta E_i^{t+1} = \frac{2}{3}(1-f)(1 + \Delta E^t) \quad (9)$$

where we set in Eq. (6) all $z(\delta) = 1/3$. At the critical point we expect all energies just to be sufficient to go over the barrier; hence set $\Delta E_i^{t+1} = \Delta E^t = E_b^c$ in Eq. (9). Solving the resulting equation yields

$$E_b^c = \frac{2(1-f)}{1+2f} \quad (10)$$

In Fig. 4 this rough estimate of the transition line is shown as a dotted line.

This simple calculation captures the physics of the problem. However, it is easy to improve it in the following way. As before, we assume the energy of toppling grains to be distributed equally among the three neighbors of the subsequent row. However, we no longer assume all active sites to carry the *same* energy, instead we compute the energy profile at the edge of a cluster. To this end let us consider a semi-infinite cluster with

$S'_i = 1$ for $i \leq 0$ and $S'_i = 0$ for $i > 0$. According to Eq. (6), we are looking for a stationary solution of the equation of motion

$$\Delta E'_i = \frac{1-f}{3} \begin{cases} 3 + \Delta E'_{i-1} + \Delta E'_i + \Delta E'_{i+1} & \text{if } i < 0 \\ 2 + \Delta E'_{-1} + \Delta E'_0 & \text{if } i = 0 \\ 1 + \Delta E'_0 & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases} \quad (11)$$

where $\Delta E'_0 = E_b^c$. Notice that under these conditions the energy $\Delta E'_{i+1}$ does not suffice to active the grain at site $i = 1$. Solving Eq. (11) we obtain the stationary solution

$$\Delta E_i^{\text{stat}} = E_{\text{bulk}} - E_{\text{gap}} \exp(ai), \quad (i \leq 0) \quad (12)$$

where

$$\begin{aligned} E_{\text{bulk}} &= (1-f)/f \\ E_{\text{gap}} &= \frac{2+f - \sqrt{12f-3f^2}}{2f(1-f)} \\ a &= \text{arccosh} \frac{2+f}{2-2f} \end{aligned} \quad (13)$$

Thus, the critical threshold is given by the expression

$$E_b^c = \frac{2f^2 - 5f + \sqrt{12f-3f^2}}{2f(f-1)} \quad (14)$$

which slightly improves the mean field result (10), especially for small values of f (see dashed line in Fig. 4). The energy profile decreases at the edges of the cluster and saturates in the bulk at E_{bulk} , as shown in Fig. 5.

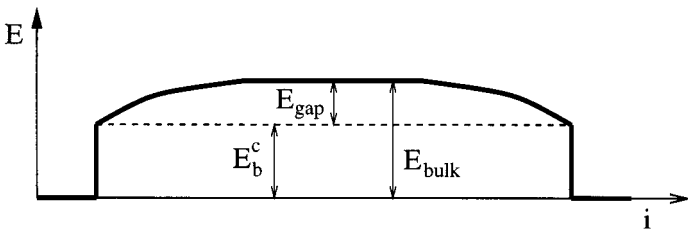


Fig. 5. Schematic drawing of the energy profile of a compact cluster in the improved mean field approximation.

The connection of our model to the experimental conditions is based on the assumption that the tilt angle of the experiment tunes the ratio between the barrier height and the difference of potential energies between two rows. If the system has been prepared at some φ_0 , we raise the tilt angle to φ ; perturbing the system in this region of mixed stability will generate an avalanche.

That is, for $\varphi > \varphi_0$ we have $E_b < E_b^c$. As the tilt angle is reduced, the size of E_b (measured in units of the potential difference) increases, until it reaches its critical value precisely at φ_0 . Thus increasing E_b in the model corresponds to lowering the tilt angle towards the value at which the system has been prepared and, as such, is precisely the boundary of dynamic stability.

Hence to reproduce the experiment we were looking for

1. fairly compact triangular regions of activation for $E_b < E_b^c$,
2. a varying opening angle of these triangles which should go to zero as E_b approaches E_b^c from below.

The number of “time steps” that correspond to the DD experiment can be estimated as the number of rows of beads from top to bottom of the plate, i.e., about 3000.

We simulated the model defined in Eqs. (6)–(8) to check whether it is possible to reproduce the qualitative features of the experiment. Indeed we found this to be the case, as can be seen in Fig. 6. The two avalanches were

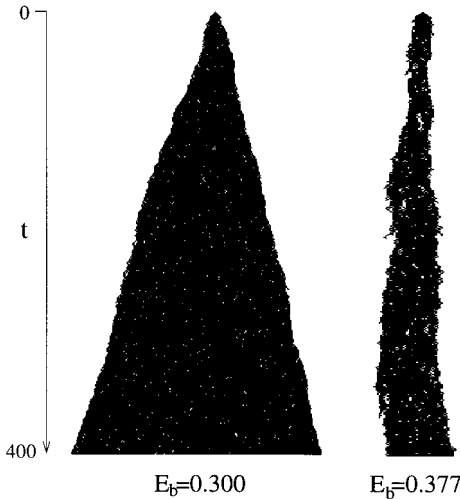


Fig. 6. Typical avalanches starting from a single seed with dissipation $f = 0.5$ far away and close to criticality.

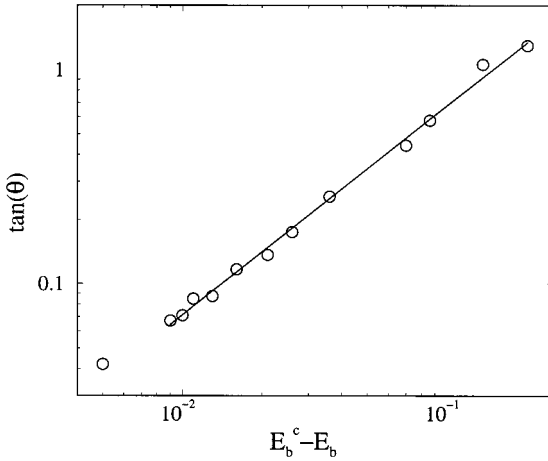


Fig. 7. Opening angle θ of activated triangular regions as a function of the distance from criticality in a double-logarithmic representation.

produced for dissipation $f = 0.5$, activating a single site at $t = 0$, to which an initial energy of $E_0 = 500$ was assigned.⁶ The avalanches were compact, triangular, and with fairly straight edges. The edges became rough only when E_b was very close to its critical value, as can be seen on the right hand side of Fig. 6. The opening angle of the active regions θ decreased as E_b increased towards E_b^c , which is shown in Fig. 7. From these simulations we obtain the estimate (see Eq. (4))

$$x = v_{\parallel} - v_{\perp} = 0.98(5) \simeq 1 \quad (15)$$

We conclude that measuring the dependence of the avalanche opening angle on $\Delta\varphi$ in the experiment should also give a linear law. This prediction is in agreement with the data presented by DD.^(4, 5)

Furthermore, the density of active sites in the interior of the triangular regions is found to be almost constant, indicating a first-order transition. These results suggest that the transition belongs to the CDP universality class, which is characterized by the critical exponents⁽¹⁰⁾

$$v_{\parallel} = 2, \quad v_{\perp} = 1, \quad \beta = 0 \quad (16)$$

These observations pose, however, a puzzle: since we believe that DP is the generic situation, we would expect to find non-compact active regions and

⁶ Note that after less than 20 time steps all the initial energy has been dissipated.

DP exponents. In the following section we present a careful numerical analysis of the critical behavior of our model which resolves this problem: the exponents seen in our simulations (and in the experiment) should cross over to the DP values, but only if one gets very deep into the critical region.

V. CROSSOVER TO DIRECTED PERCOLATION

The linear law observed in Fig. 7 can be explained by assuming compact clusters whose temporal evolution is determined by the fluctuations of their boundaries. The boundaries perform an effective random walk with a spatial bias proportional to $E_b - E_b^c$. Therefore, the critical model should behave in the same way as a Glauber–Ising model at zero temperature, i.e., the transition should belong to the CDP universality class. However, according to the DP conjecture⁽¹¹⁾ any continuous spreading transition from a *fluctuating* active phase into a single frozen state should belong to the universality class of directed percolation (DP), provided that the model is defined by short range interactions without exceptional properties such as higher symmetries or quenched randomness (see Section VI). Clearly, the present model fulfills these requirements. It has indeed a fluctuating active state and exhibits a phase transition into a single absorbing state which is characterized by a positive one-component order parameter. According to these arguments, the phase transition should belong to the DP universality class.

In order to understand this apparent paradox we perform high-precision Monte-Carlo simulations for dissipation $f = 0.5$. We employ time-dependent simulations,⁽¹²⁾ i.e., we topple a single grain in the center and analyze the properties of the resulting cluster. As usual for this type of simulations, we measure the survival probability $P(t)$, the number of active sites $N(t)$, and the mean square spreading from the origin $R^2(t)$ averaged over the surviving runs. At criticality, these quantities are expected to show an asymptotic power law behavior

$$P(t) \sim t^{-\delta}, \quad N(t) \sim t^\eta, \quad R^2(t) \sim t^{2/z} \quad (17)$$

where δ , η , and z are critical exponents which label the universality class. In the case of CDP these exponents are given by refs. 7 and 10

$$\delta = 1/2, \quad \eta = 0, \quad z = 2 \quad (18)$$

whereas DP is characterized by the exponents⁽²⁾

$$\delta = 0.1595, \quad \eta = 0.3137, \quad z = 1.5807 \quad (19)$$

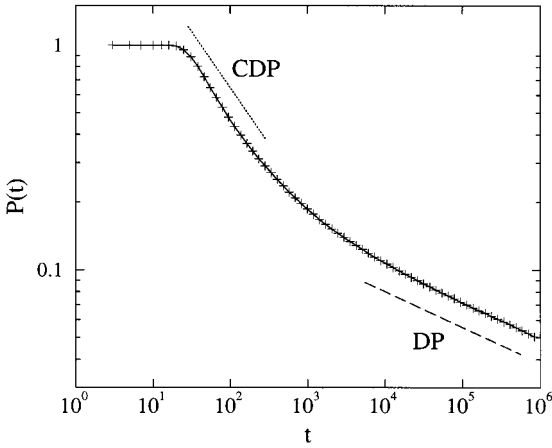


Fig. 8. Mean survival probability $P(t)$ of the toppling process at criticality averaged over 50,000 independent runs. The predicted slopes for CDP and DP are indicated by dotted and dashed lines, respectively.

In order to eliminate finite-size effects, we use a dynamically generated lattice adjusted to the actual size of the cluster. Moreover, we observe that the initial non-universal transient is minimal if an excitation energy $E_0 \simeq 15$ is used. Detecting deviations from power-law behavior in the long-time limit we estimate the critical energy by $E_b^c = 0.385997(5)$. Our numerical results (obtained from simulations at the critical point) are shown in Figs. 8–10. In all measurements we observe different temporal regimes:

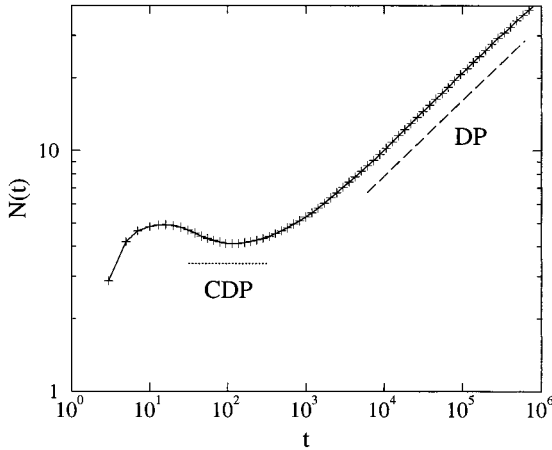


Fig. 9. Average number of active sites $N(t)$. The expected slopes are indicated in the same way as in Fig. 8.

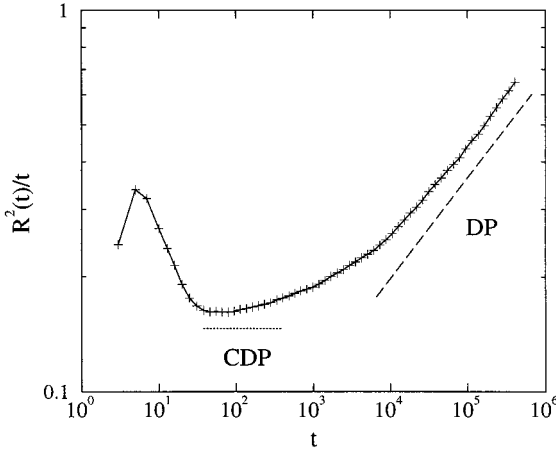


Fig. 10. Mean square spreading from the origin averaged over surviving runs. In order to demonstrate the crossover from CDP to DP we divided $R^2(t)$ by t . The expected slopes $(2-z)/z$ are indicated by dotted and dashed lines.

1. During the first few time steps, the activation energy is distributed to the nearest neighbors whereby the cluster grows at maximal speed. Therefore, the survival probability $P(t)$ is 1 and the particle number $N(t)$ grows linearly.

2. In the intermediate regime, which extends up to a few hundred time steps, the inactive islands within the cluster are not yet able to break up the cluster into separate parts. Thus, the cluster can be considered as being compact and the temporal evolution is governed by a random walk of its boundaries. In this regime we observe a power-law behavior with CDP exponents (indicated by dotted lines in Figs. 8–10).

3. The intermediate regime is followed by a long crossover from CDP to DP extending over almost two decades up to more than 10^4 time steps.⁷

4. Finally the system enters an asymptotic DP regime (indicated by dashed lines in Figs. 8–10).

The crossover from CDP to DP is illustrated in Fig. 11. Two avalanches are plotted on different scales. The left one represents a typical avalanche within the first few thousand time steps. As can be seen, the cluster appears

⁷ Note that the crossover in the present model is different from the one studied in ref. 13, where inhomogeneous interactions at the cluster's boundaries were assumed.

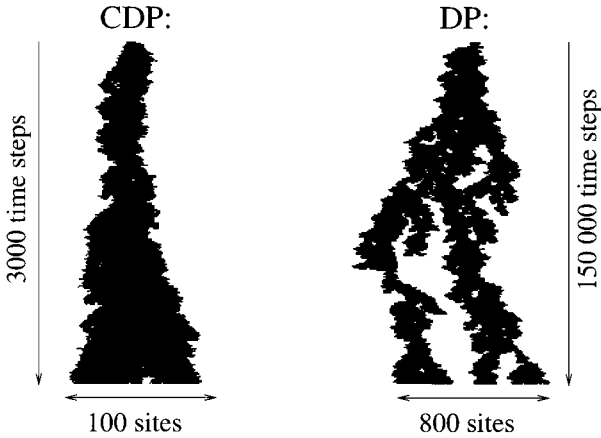


Fig. 11. Typical clusters generated at criticality on small and large scales, illustrating the crossover from CDP to DP.

to be compact on a lateral scale up to 100 lattice sites. However, as shown in the right panel of Fig. 11, after a very long time the cluster breaks up into several branches. The right hand figure shows a typical cluster on a scale of 150,000 time steps, where the branches still have a certain characteristic thickness. Going to even larger scales the width of the branches becomes irrelevant and we obtain the typical patterns of critical DP clusters.

In comparison with ordinary DP lattice models, in the present model the observed crossover is unusually slow. This due to short-range correlations between active sites leading to active branches with a certain typical thickness ξ_{act} . In ordinary DP lattice models the average size of active branches is of the order of a few lattice spacings. In the present case, however, we find a much larger value $\xi_{\text{act}} \approx 20$.

Based on this observation, the typical crossover time t_c can be approximated as follows. In order to cross over to DP, the average size of *inactive* regions between neighboring branches ξ_{inact} has to become larger than the thickness of the branches ξ_{act} . In Fig. 12 we plot both quantities as a function of time at criticality, using a lattice with $N=2^{14}$ sites and homogeneous initial conditions $E_i^{t=0} = 2$. Initially $\xi_{\text{act}} = N$ and $\xi_{\text{inact}} = 0$. As time evolves, the average size of active branches decreases and saturates at a constant value $\xi_{\text{act}} \approx 20$. However, the average size of inactive regions ξ_{inact} continues to grow and exceeds ξ_{act} at time $t_c \approx 10^5$. As can be seen, this provides a good estimate of the typical time where the critical behavior of the system crosses over to DP.

In order to observe the crossover experimentally, it would be interesting to know how the crossover time t_c can be reduced. To this end we

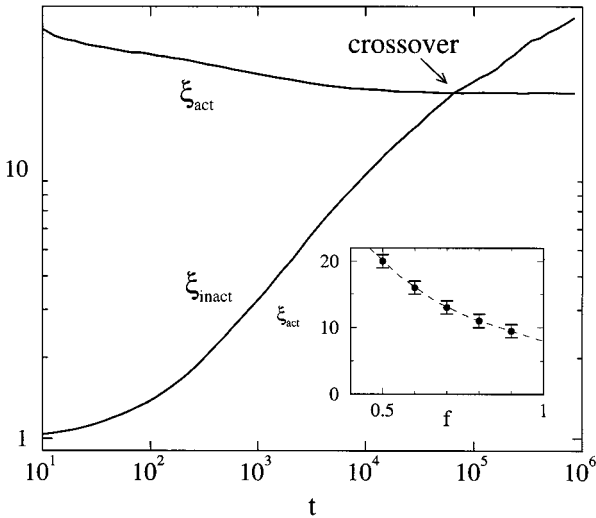


Fig. 12. Mean sizes of active and inactive regions as a function of time, starting from homogeneous initial conditions with dissipation $f = 0.5$ (see text). The inset shows the saturation value of ξ_{act} as a function of the dissipation f .

measure ξ_{act} for several values of the dissipation f (see inset of Fig. 11). It turns out that by increasing f the typical size of active branches can be decreased down to 10 lattice spacings. Consequently, the crossover time can be reduced by more than one decade. Hence, for an experimental verification of DP, systems with high dissipation are more appropriate.

The influence of the dissipation can easily be explained within the improved mean field approximation of Section IV. Clearly, the stability of a cluster against breakup into several branches by fluctuations depends on the energy gap $E_{gap} = E_{bulk} - E_c$. As can easily be verified, this energy difference (and therewith the stability of compact clusters) decreases with increasing dissipation f , explaining the observed f -dependence.

VI. THE EFFECT OF RANDOMNESS

The above model describes the physics of flowing sand in a highly idealized manner. In particular, it ignores the fact that spreading avalanches may be subjected to frozen disorder. For example, irregularities of the plate and the velvet cloth could lead to quenched randomness in the equations of motion. Moreover, the system prepares itself in an initial state which is not fully homogeneous. Thus, we have to address the question to what extent quenched randomness will affect the expected crossover to DP.

Certain types of quenched disorder are known to change the critical behavior of DP. For example, Moreira and Dickman studied the diluted contact process with *spatially* quenched disorder.⁽¹⁴⁾ Even for small amplitudes quenched randomness was found to destroy the DP transition, turning algebraic into logarithmic laws. Janssen⁽¹⁵⁾ confirmed and substantiated these findings by a field-theoretic analysis. Recently Cafiero *et al.*⁽¹⁶⁾ mapped DP with spatially quenched disorder onto a non-markovian process with memory exhibiting the same nonuniversal properties. The memory is due to the formation of bound states of particles in those regions where the percolation probability is very high. As shown by Webman *et al.*, these bound states give rise to a glassy phase separating active and inactive parts of the phase diagram.⁽¹⁷⁾ Similar nonuniversal properties were also observed in DP processes with *temporally* quenched disorder.⁽¹⁸⁾

In all cases investigated so far, quenched disorder destroys the DP transition. However, the disorder in the DD experiment is different in nature. Clearly, it is neither spatially nor temporally quenched, rather it depends on both space and time. On the level of our model we may think of randomly varying energy barriers

$$E_b \rightarrow E_b + A\eta(x, t) \quad (20)$$

where the amplitude A controls the intensity of disorder. Here $\eta(x, t)$ is a white Gaussian noise specified by the correlations

$$\overline{\eta(x, t) \eta(x', t')} = \delta^d(x - x') \delta(t - t') \quad (21)$$

where $d=1$ denotes the spatial dimension. In the standard situation of quenched noise of this type $\eta(x, t)$ is kept fixed while the experiment is repeated and the quantities under investigation are averaged over many independent avalanches. Yet in the DD experiment, the situation is different. Here once the sand has been poured, a particular realization of the random variables has been selected. However, there is no process to repeat the experiment over and over again with a fixed $\eta(x, t)$. Rather, after each avalanche the system is prepared again (by pouring sand or by starting an avalanche elsewhere). Hence the averaging process is done simultaneously over the $\eta(x, t)$ and the stochastic dynamic process that generates the avalanches. This type of averaging is of the annealed type and therefore less likely to alter the critical behavior than its quenched version.

In order to find out whether fully quenched disorder affects the asymptotic critical behavior of DP, we simulated a directed bond percolation process with randomly distributed bond probabilities between p^* and 1. For $p^* = 0.289(1)$, we find asymptotic power laws with DP exponents, indicating that the transition is not affected by spatio-temporally quenched

noise. Therefore, we expect the same to be true in the case of annealed disorder in our model for flowing sand.

To support this point of view, we study the case of quenched randomness in the DP Langevin equation⁽¹⁹⁾

$$\partial_t \rho(x, t) = a\rho(x, t) - g\rho^2(x, t) + D \nabla \rho(x, t) + \Gamma \sqrt{\rho(x, t)} \xi(x, t) \quad (23)$$

where $\rho(x, t)$ is the particle density and a represents the percolation probability. $\xi(x, t)$ is a Gaussian white noise which represents the intrinsic randomness of the DP process. At the critical dimension $d=4$, where fluctuations start to contribute, the Langevin equation (22) is invariant under scaling transformations $x \rightarrow bx$, $t \rightarrow b^2t$, and $\rho \rightarrow b^{-2}\rho$.

In order to include spatio-temporally quenched randomness, we allow for small variations of a , i.e., we add the term

$$A\rho(x, t) \eta(x, t)$$

on the right hand side of Eq. (22). However, as can be shown by simple dimensional analysis, this term is *irrelevant* in $d=4$ dimensions, i.e., it decreases and eventually vanishes under scaling transformations. This observation strongly supports the result that the DP transition in our model is indeed not affected by quenched randomness.

We emphasize that the irrelevance of quenched randomness in our model is due to the special role of “time” which coincides with the vertical coordinate of the plane. That is, for each time step the stochastic processes take place in a different random environment. To that extent the DD experiment differs from other DP-related experiments such as catalytic reactions where spatially quenched disorder affects the critical behavior.

The arguments given above assume that the quenched noise $\eta(x, t)$ does not involve long-range correlations due to a “memory” of rolling grains. This is supported by recent studies of a single rolling grain on an inclined rough plane.⁽²⁰⁾ By means of molecular dynamics simulations it was shown that the motion of a rolling grain consists of many small bounces on each grain of the supporting layer. Therefore, the rolling grain quickly dissipates almost all of the energy gain from the previous step and thus forgets its history very fast. For this reason it seems to be unlikely that quenched disorder of the prepared layer involves long-range correlations. However, this prediction should be verified experimentally.

VII. CONCLUSIONS

We introduced a simple model for flowing sand on an inclined plane. The model is inspired by recent experiments and reproduces some of the

observed features. In contrast to the experiment, which prepares itself in a self-organized critical state, our model needs to be tuned to a critical point by varying the energy barrier E_b . At criticality the system undergoes a non-equilibrium phase transition from an inactive (dry) phase with finite avalanches to an active (wet) phase where the mean size of avalanches diverges. Analyzing the critical behavior near the transition, we obtained the following results:

1. On short scales, i.e., on scales considered in the DD experiment, the model reproduces the experimentally observed triangular compact avalanches. In the active phase their opening angle θ is predicted to vary linearly with $\Delta\varphi$. This prediction has to be compared with the model proposed by Bouchaud and Cates⁽²¹⁾ where θ was found to vary as $\sqrt{\Delta\varphi}$.

2. On very large scales the critical behavior of the model crosses over to ordinary DP. Thus, the DD experiment could serve as a first physical realization of directed percolation. Crossover to DP is seen in the model after about 10^4 time steps, whereas the DD experiment stops at about 3000 steps (i.e., rows of beads). Hence in order to observe the crossover in the experiment, larger system sizes and/or smaller beads would be required.

3. We have shown that quenched randomness with short-range correlations due to irregularities in the experiment should not affect the asymptotic critical behavior.

4. The typical time needed to cross over to DP is found to decrease with increasing dissipation.

Thus, in order to create experimental conditions favoring a crossover to DP, we suggest to use small glass beads, large system sizes, and an initial angle φ_0 where the dissipation of energy per toppling grain is maximal. For physical reasons we would expect the dissipation to be maximal for small angles φ_0 , but this has to be verified in the actual experiment.

As a necessary precondition for a crossover to DP, compact clusters must be able to split up into several branches, as illustrated in Fig. 11. Thus, before measuring critical exponents, this feature has to be tested experimentally. To this end the DD experiment should be performed repeatedly at the critical tilt $\varphi = \varphi_0$. In most cases the avalanches will be small and compact. However, large avalanches, reaching the bottom of the plate, will sometimes be generated. If these avalanches are non-compact (consisting of several branches) we expect the asymptotic critical behavior to be described by DP. Only then it is worthwhile to optimize the experimental setup and to measure the critical exponents quantitatively.⁸

⁸ Note added in proof: After submission of this article, DD observed splitting avalanches for very high angles φ_0 . However, the branches do not propagate uniformly, i.e., they may have different velocities.

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